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The Lee identities in Topoi, I[1]

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Abstract

We investigate conditions under which various topoi satisfy equations, known as the Lee identities, that are similar to and weaker than De Morgan's law. © 1997 Elsevier Science B.V.

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1. Introduction

This article is concerned with characterizing topoi whose truth value objects (as Heyting algebras) have special algebraic properties; we explain this more precisely later.

The algebraic structures that we consider are pseudocomplemented distributive lattices with 0 and 1. The equational classes of pseudocomplemented distributive lattices with 0 and 1 form an ω-chain: $\mathbf{B}_{-1} \subsetneq \mathbf{B}_0 \subsetneq \mathbf{B}_1 \subsetneq \cdots \subsetneq \mathbf{B}_\omega$, where $\mathbf{B}_{-1}$ contains only the trivial algebra with $0 = 1$, $\mathbf{B}_0$ is the class of Boolean algebras and $\mathbf{B}_1$ is the class of Stone algebras that are algebras satisfying De Morgan's law; by an algebraic property, we mean a defining equation I_n' for such a class $\mathbf{B}_n$. We note that every Heyting algebra is a particular kind of pseudocomplemented distributive lattice with 0 and 1.

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The specific examples of pseudocomplemented distributive lattices with 0 and 1 that we consider are the following:

(1) Open(X), which is the Heyting algebra of open sets of a topological space X, and

(2) $\Omega_{\mathscr{E}}$, which is the Heyting algebra of truth-values of an arbitrary topos $\mathscr{E}$.

It should be noted that the axioms characterizing $\mathbf{B}_n$, where $n \geq 2$, are weaker than De Morgan's law. Other non-equational conditions, weaker than De Morgan's law have been studied by Johnstone [9]. Conditions under which the open set lattice of a topological space satisfies De Morgan's law or I_1' were investigated by Gleason [5]. Conditions under which the truth-value object of a topos satisfies I_1', were investigated by Johnstone [9, 10].

We now briefly describe the contents of the other sections. In Section 2 we collect the information about distributive lattices needed later, while in Section 3 we investigate the consequences of the validity of the equational axioms in various kinds of topoi. We characterize topological spaces whose open-set lattices satisfy I_n' for some given n, thus generalizing Gleason's result on projective topological spaces [5]. We also relate the internal validity of the equation I_n' for some n, in arbitrary topoi, to properties of maximal ideals in rings and distributive lattices in these topoi, generalizing earlier results of Johnstone [7]. We list some open questions at the end of Section 3. We end the introduction by listing some notational conventions that we have adopted.

Notational conventions

(1) Given integers m and n, $m..n$ denotes the set of integers i such that $m \leq i \leq n$. Thus $i \in m..n$ means that i is an integer such that $m \leq i \leq n$.

(2) Given a set S, Fin(S), $\mathscr{P}(S)$ and $\#(S)$ denote, respectively, the set of finite subsets of S, the power-set of S and the cardinality of S.

(3) For all sets A, B such that B is included in A we denote by ι_A, the identity function on A, and by $\iota_{B \subset A}$, the inclusion mapping of B in A.

(4) Given a function $f: D \to C$, a subset D' of D and a superset C' of C, we define $f|_{D'} := f \circ \iota_{D' \subset D}$, $f|^{C'} := \iota_{C \subset C'} \circ f$, $f_{>} : \mathscr{P}(D) \to \mathscr{P}(C) := S \mapsto \{f(s) \in C \mid s \in S\}$, and $f^{<} : \mathscr{P}(C) \to \mathscr{P}(D) := T \mapsto \{s \in D \mid f(s) \in T\}$.

(5) In any category $\mathscr{C}$, we have the maps Dom and Cod that send a morphism to its domain and codomain respectively. Thus $\mathrm{Dom}^{<}$ and $\mathrm{Cod}^{<}$ are defined by (4).

(6) In any category $\mathscr{C}$, given a morphisms f, we define $\mathrm{Lcomp}_f : \mathrm{Cod}^{<}(\mathrm{Dom}(f)) \to \mathrm{Cod}^{<}(\mathrm{Cod}(f)) := g \mapsto f \circ g$. We shall have occasion to use $(\mathrm{Lcomp}_f)^{<} : \mathscr{P}(\mathrm{Cod}^{<}(\mathrm{Cod}(f))) \to \mathscr{P}(\mathrm{Cod}^{<}(\mathrm{Dom}(f))$.

We note that given a family $F \in \mathscr{P}(\mathrm{Cod}^{<}(\mathrm{Cod}(f)))$ of morphisms with common codomain, the codomain of f, $(\mathrm{Lcomp}_f)^{<}(F) = \{g \in \mathrm{Cod}^{<}(\mathrm{Dom}(f) \mid f \circ g \in F\}$.

(7) Boldface letters are used to denote finite sequences; the ith term of $\mathbf{a}$ is a_i. Universal quantification over a set of variable $\{x_i \mid i \in 1..r\}$ is denoted either by $\forall x_1 \ldots \forall x_r$ or by $\forall x_1, \ldots, x_r$. Similar remarks apply to existential quantification.

(8) Given a category $\mathscr{C}$, $\mathrm{Op}(\mathscr{C})$ denotes the opposite category. $\mathrm{Obj}(\mathscr{C})$ and $\mathrm{Mor}(\mathscr{C})$ denote, respectively, the collection of objects of $\mathscr{C}$ and the collection of morphisms of $\mathscr{C}$.

2. Distributive lattices

In this section we gather for later use all the results on distributive lattices that we shall need.

Definition 2.1. $D_{01} :=$ the class of distributive lattices with 0 and 1 in the language with symbol-set $\{0, 1, \wedge, \vee, \leq\}$. $a \to b :=$ the *relative pseudocomplement of a with respect to b*, which, if it exists, is the largest element $c \in L$ such that $a \wedge c \leq b$. $a^* :=$ the *pseudocomplement of a*, which, if it exists, is $a \to 0$.

Definition 2.2. Given $L \in D_{01}$, if for every $a \in L$, a^* exists, then L is said to be *pseudocomplemented.*

Definition 2.3. Given $L \in D_{01}$, if for all $a, b \in L$, $a \to b$ exists, then L is called a *Heyting algebra*. For the purposes of this article, we shall consider Heyting algebras to be particular kinds of pseudocomplemented distributive lattices.

Definition 2.4. Pseudocomplemented lattices, regarded as structures for the language with symbol-set $\{0, 1, \wedge, \vee, \leq, *\}$ are called *distributive p-algebras.*

Definition 2.5. $\mathbf{B}_\omega :=$ the class of distributive p-algebras.

Remark 2.6. $\mathbf{B}_\omega$ is the equational class axiomatized by the axioms of D_{01} together with the following axioms [19]:

(0) $0^* = 1$. (1) $1^* = 0$. (3) $\forall x, y(x \wedge (x \wedge y)^* = x \wedge y^*)$.

Remark 2.7. The equational subclasses of $\mathbf{B}_\omega$ are exactly $\mathbf{B}_{-1} \subsetneq \mathbf{B}_0 \subsetneq \mathbf{B}_1 \subsetneq \cdots \subsetneq \mathbf{B}_\omega$, where $\mathbf{B}_{-1}$ contains only the trivial p-algebra, $\mathbf{B}_0$ is the class of Boolean algebras and $\mathbf{B}_1$ is the class of Stone algebras. $\mathbf{B}_0$ is axiomatized by the axioms for distributive p-algebras and $\forall x(x \vee x^* = 1)$. $\mathbf{B}_1$ is axiomatized by the axioms for distributive p-algebras and $\forall x(x^* \vee x^{**} = 1)$. In general for all $r \geq 1$, $\mathbf{B}_r$ is axiomatized by the axiom I_r, defined as follows, in addition to the axioms for distributive p-algebras [6]:

$$I_r := \forall x_0 \forall x_1 \cdots \forall x_r \left(\bigwedge_{i,j \in 0..r, i<j} (x_i \wedge x_j = 0) \to \left(\bigvee_{i \in 0..r} x_i^* = 1 \right) \right).$$

I_r may be rewritten as an equation I_r', namely,

$$I_r' := \forall x_1 \cdots \forall x_r \left(\left(\left(\bigwedge_{i \in 1..r} x_i \right)^* \vee \left(\bigvee_{i \in 1..r} \left(\left(\bigwedge_{j \in 1..r \setminus \{i\}} x_j \right) \wedge x_i^* \right)^* \right) \right) = 1 \right).$$

If the Heyting algebra of open sets of a topological space satisfies I_r $(r \geq 2)$, then we call it an *r-Lee space*.

3. The Lee identities in topoi

The truth-value object of a topos $\mathcal{T}$, denoted by $\Omega_{\mathcal{T}}$, is internally a Heyting algebra and hence a distributive p-algebra [8, pp. 137–138]. In this section we study conditions under which $\Omega_{\mathcal{T}} \in \mathbf{B}_n$, i.e., $\Omega_{\mathcal{T}}$ satisfies the Lee identity I_n for some $n \in \omega$. We shall derive general results for arbitrary topoi and also work out the details in certain special cases of particular interest. These are as follows:

(1) $\mathcal{T} = \mathcal{S}^{\mathrm{Op}(\mathcal{C})}$. The special cases in which $\mathcal{C}$ is a monoid and in which $\mathcal{C}$ is a poset will follow as corollaries. The cases $n = 0$ and $n = 1$ have been studied by Johnstone [7].

(2) $\mathcal{T} = \mathrm{Shv}(X)$, where X is a topological space.

It should be noted that the results for (2) may be stated directly in terms of the lattice of open sets of X, Open(X), as Open(X) is isomorphic to the global sections of $\Omega_{\mathrm{Shv}(X)}$. Indeed we shall derive some of the results using purely topological methods and state them in the context of topological spaces. For instance, the following are well-known [4, p. 22]:

(1) The following are equivalent for a T_2 space:

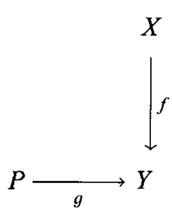
 (a) Open(X) satisfies I_0.

 (b) Open(X) is a Boolean algebra.

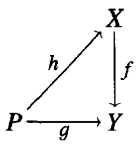
 (c) X is discrete.

(2) Open(X) satisfies $I_1 \Leftrightarrow X$ is extremally disconnected.

Extremally disconnected spaces have been studied in the context of functional analysis by several authors [4, p. 22]. They were characterized from the category-theoretic point of view by Gleason, who proved the following theorems [5]. We shall follow Johnstone's formulation of these theorems and their generalizations [12]. We recall following [12] that an object P in a category $\mathcal{C}$ is said to be projective if and only if for every diagram of the form



with f an epimorphism in $\mathscr{C}$ may be completed to one of the form



i.e. there exists $h: P \to X$ such that $fh = g$. More generally, one may consider E-projectives or projectives with respect to a class E of morphisms for which the morphism f in the diagram above is required to belong to a particular class of epimorphisms E (for instance, the regular epimorphisms).

Theorem 3.1. *In the category of compact T_2 spaces and continuous maps, the projective objects are precisely the extremally disconnected spaces.*

Theorem 3.2. *For any compact T_2 space X, there is a continuous surjection (called the Gleason-cover map) $e: \gamma X \to X$, where γX is compact, T_2 and extremally disconnected, which is "minimal" in the sense that every other such surjection factors surjectively through e. Moreover, this property characterizes γX up to (unique) homeomorphism in the category of spaces over X.*

These results have since been extended by several authors, in general, by enlarging the category under consideration [13, pp. 104–105]. Theorems 3.11 and 3.15 are analogues of these results for $n \geq 2$.

Johnstone has studied the general case for $n = 1$. He proved the following theorem [10].

Theorem 3.3. *The following conditions on a topos $\mathscr{T}$ are equivalent:*
(1) $\mathscr{T} \models I_1$, *i.e. De Morgan's law holds in* $\mathscr{T}$.
(2) *If R is a commutative ring in $\mathscr{T}$, then every maximal ideal of R is prime.*
(3) *Same statement as* (2) *for distributive lattices.*
(4) *Same statement as* (2) *for Boolean algebras.*

Theorem 3.28 is a generalization of this result for $n \geq 2$.

We begin by proving an analogue of Theorem 3.1 for Lee identities in the category *CHaus of compact T_2 spaces and continuous maps*. X, the condition $\mathrm{Shv}(X) \models I_n$ is equivalent to $\mathrm{Open}(X) \models I_n$ [9]. In what follows, $\mathrm{Closed}(X)$ denotes the set of closed sets of a topological space X and $\mathrm{Cl}(A)$ denotes the closure of a set A. Exactly how Theorem 3.2 generalizes for $n \geq 2$ is currently being investigated.

Definition 3.4. A continuous surjection $\rho: E \to A$ is said to be *minimal*: $\Leftrightarrow$ for every $F \in \mathrm{Closed}(E) \setminus \{E\}$, $\rho_{>}(F) \subsetneq A$.

Definition 3.5. For every $O \in \mathrm{Open}(E)$ and for every $\rho: E \to A, \forall_\rho(O) := \{y \in A \mid \rho^{<}(\{y\}) \subset O\}$.

The following lemma lists some basic facts about $\forall_\rho$.

Lemma 3.6. *Let the continuous surjection $\rho : E \to A$ be given. Then* (1) $\forall_\rho(O) = A\backslash\rho_>(E\backslash O)$, (2) $\forall_\rho(\emptyset) = \emptyset$ *if ρ is surjective, and* (3) $\forall O_1, O_2 \in \mathrm{Open}(E)$, $\forall_\rho(O_1) \cap \forall_\rho(O_2) = \forall_\rho(O_1 \cap O_2)$.

Proposition 3.7. *Let the spaces A, E and a continuous surjection $\rho : E \to A$ be given. Then ρ is minimal $\Leftrightarrow$ for every $O \in \mathrm{Open}(E)$, $\rho_>(O) \subset \mathrm{Cl}(\forall_\rho(O))$.*

Proof. ($\Rightarrow$): [5].

($\Leftarrow$): Assume that for every $O \in \mathrm{Open}(E)$, $\rho_>(O) \subset \mathrm{Cl}(\forall_\rho(O))$, and that ρ is not minimal. Hence we may choose $E' \in \mathrm{Closed}(E)\backslash\{E\}$ such that $A = \rho_>(E')$. Set $O := E\backslash E'$. By assumption, $O \neq \emptyset$. Hence $\rho_>(O) \neq \emptyset$. But $\rho_>(O) \subset \mathrm{Cl}(A\backslash\rho_>(E\backslash O)) = \mathrm{Cl}(A\backslash\rho_>(E')) = \mathrm{Cl}(A\backslash A) = \mathrm{Cl}(\emptyset) = \emptyset$, which is a contradiction. Hence the result. □

Lemma 3.8. *X is an n-Lee space $\Leftrightarrow$ for every family $(O_i \mid i \in 1..(n+1))$ of $n+1$ pairwise disjoint open sets in X, $\bigcap_{i\in 1..(n+1)} \mathrm{Cl}(O_i) = \emptyset$.*

Proof. X is an n-Lee space

$\Leftrightarrow$ for every family as in the preceding, $\bigcup_{i\in 1..(n+1)} \mathrm{Int}(X\backslash O_i) = X$

$\Leftrightarrow$ for every family as in the preceding, $\bigcap_{i\in 1..(n+1)} X\backslash\mathrm{Int}(X\backslash O_i) = \emptyset$

$\Leftrightarrow$ for every family as in the preceding, $\bigcap_{i\in 1..(n+1)} \mathrm{Cl}(O_i) = \emptyset$.

Proposition 3.9. *Let a continuous surjection of T_2 spaces $\rho : E \to A$ be given. Assume that A is an n-Lee space and that ρ is minimal. Then ρ is at most n to 1.*

Proof. We assume, towards a contradiction, that we may choose $a \in A$ and $n+1$ distinct points $x_1, x_2, \dots, x_{n+1} \in E$ such that for every $i \in 1..(n+1)$, $\rho(x_i) = a$. As E is a T_2 space, we may choose for all $i \in 1..(n+1)$ pairwise disjoint open neighbourhoods $O_i \in \mathrm{Open}(E)$ of the points x_i. Then for every $i \in 1..(n+1)$, $E\backslash O_i$ is closed and (as E is compact), compact. Hence for every $i \in 1..(n+1)$, $\rho_>(E\backslash O_i)$ is compact and hence closed. Hence for every $i \in 1..(n+1)$, $\forall_\rho(O_i) = A\backslash\rho_>(E\backslash O_i) \in \mathrm{Open}(A)$. For distinct $i, j \in 1..(n+1)$, $\forall_\rho(O_i) \cap \forall_\rho(O_j) = \forall_\rho(O_i \cap O_j) = \forall_\rho(\emptyset) = \emptyset$ (by Lemma 3.6).

Hence $(\forall_\rho(O_i) \mid i \in 1..(n+1))$ is a collection of $n+1$ pairwise disjoint open sets. As A is an n-Lee space, it follows from Lemma 3.8 that

$$\bigcap_{i\in 1..(n+1)} \mathrm{Cl}(\forall_\rho(O_i)) = \emptyset. \tag{1}$$

As ρ is minimal, it follows from Proposition 3.7 that for every $i \in 1..(n+1)$, $\rho_>(O_i) \subset \mathrm{Cl}(\forall_\rho(O_i))$. Hence for every $i \in 1..(n+1)$, $\rho(x_i) = a \in \rho_>(O_i) \subset \mathrm{Cl}(\forall_\rho(O_i))$ which contradicts (1). Hence the result. □

Lemma 3.10. *Let the compact Hausdorff spaces A, D and the continuous surjection $\rho : D \to A$ be given. Then there exists $E \in \mathrm{Closed}(D)$ such that $\rho_>(E) = A$, but for every $F \in \mathrm{Closed}(E)\backslash\{E\}$, $\rho_>(F) \subsetneq A$, i.e., $\rho|_E : E \to A$ is minimal* [13].

Theorem 3.11. *The following are equivalent for every compact T_2 space A:*

(1) *A is an n-Lee space.*

(2) *The Gleason-cover map $e : \gamma A \to A$ is at most n to 1. (Note that e is minimal.)*

(3) *For every compact T_2 space E and every continuous surjection $\rho : E \to A$ (ρ is minimal $\Rightarrow \rho$ is at most n to 1).*

(4) *For every compact T_2 space E and for every continuous surjection $\rho : E \to A$ there exists $F \in \mathrm{Closed}(E)$ such that $\rho|_F : F \to A$ is at most n to 1 and $\rho_>(F) = A$.*

(5) *Every diagram of the form*

$$\begin{array}{ccc} & & E \\ & & \downarrow h \\ A & \xrightarrow{k} & B \end{array}$$

in CHaus with h surjective can be completed to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ g \downarrow & & \downarrow h \\ A & \xrightarrow{k} & B \end{array}$$

with g at most n to 1.

Proof. We establish the following chains of implications:

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1), (3) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2).

(1) $\Rightarrow$ (2): This follows from Proposition 3.9 as e is minimal.

(2) $\Rightarrow$ (3): We prove this by contraposition. Let E, $\rho : E \to A$ be given such that ρ is minimal. We assume that (3) does not hold, i.e. ρ is not at most n to 1. Hence we may choose $n+1$ distinct points $x_1, x_2, \ldots, x_{n+1} \in E$ such that for every $i \in 1..(n+1)$, $\rho(x_i) = a \in A$.

As γA in projective in CHaus, we have the lifting

$$\begin{array}{ccc} & \overset{f}{\nearrow} & E \\ & & \downarrow \rho \\ \gamma A & \xrightarrow[e]{} & A \end{array}$$

Hence, as the preceding diagram commutes and as f is surjective, we may choose for every $i \in 1..(n+1)$, a distinct point $y_i \in f^{<}(\{x_i\}) \subset \gamma A$, such that for every $i \in 1..(n+1)$, $e(y_i) = \rho f(y_i) = \rho(x_i) = a$. Hence the Gleason-cover map e is not at most n to 1. Hence (2) does not hold. Hence the result.

(3) $\Rightarrow$ (4): Let E and $\rho : E \to A$ be given. Using Lemma 3.10 we may choose $F \in \mathrm{Closed}(E)$ such that $\rho|_F$ is minimal. Hence by (3), $\rho|_F$ is at most n to 1.

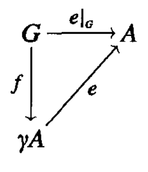
(3) ⇒ (1): We prove this by contraposition. We assume that (1) does not hold, i.e. A is not an n-Lee space. Hence we may choose a family of pairwise disjoint open sets $(O_i \mid i \in 1..(n+1))$ and $a \in A$ such that $a \in \bigcap_{i\in 1..(n+1)} \mathrm{Cl}(O_i)$. Set $X := \bigcup_{i\in 1..(n+1)} (A \setminus \bigcup_{j\in 1..(n+1)\setminus\{i\}} O_j) \times \{i\}$. $\pi : X \to A$ denotes the projection. We may, by Lemma 3.10, choose $E \in \mathrm{Closed}(X)$ such that $\rho := \pi|_E$ is minimal.

Claim. *ρ is not at most n to* 1.

Proof. Let $i \in 1..(n+1)$ be given. As ρ is surjective, $O_i \subset \rho_>(E)$. As O_i is disjoint from $\pi_>(\bigcup_{j\in 1..(n+1)\setminus\{i\}}(A \setminus \bigcup_{k\in 1..(n+1)\setminus\{j\}} O_k) \times \{j\})$, it follows that $\rho^<(O_i) \subset ((A \setminus \bigcup_{j\in 1..(n+1)\setminus\{i\}} O_j) \times \{i\})$. Hence $O_i \times \{i\} \subset ((A \setminus \bigcup_{j\in 1..(n+1)\setminus\{i\}} O_j) \times \{i\}) \bigcap E$. As $((A \setminus \bigcup_{j\in 1..(n+1)\setminus\{i\}} O_j) \times \{i\}) \cap E$ is closed for every i, it follows that for every $i \in 1..(n+1)$, $\mathrm{Cl}(O_i \times \{i\}) = \mathrm{Cl}(O_i) \times \{i\} \subset ((A \setminus \bigcup_{j\in 1..(n+1)\setminus\{i\}} O_j) \times \{i\}) \cap E$. Hence, as $a \in \bigcap_{i\in 1..(n+1)} \mathrm{Cl}(O_i)$, $\rho^<(\{a\}) = \{(a,i) \mid i \in 1..(n+1)\}$. Hence the claim. □

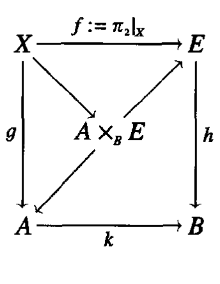
Thus $\rho : E \to A$ is minimal and yet fails to be at most n to 1. Hence (3) does not hold.

(4) ⇒ (2): Let A be given. We may, using (4), choose $F \in \mathrm{Closed}(\gamma A)$ such that $e|_F$ is surjective and at most n to 1. Using Lemma 3.10, we may choose $G \in \mathrm{Closed}(F)$ such that $e|_G$ is minimal. But using the minimality of Gleason covers we have a factorization with $e|_G = ef$, as shown in the diagram below:



As $e|_G$ is at most n to 1, so is e.

(3) ⇒ (5): Let A, E, B, k, h be given as in the hypothesis of (5). Consider the pullback $A \times_B E$, where $\pi_1 : A \times_B E \to A$ is surjective as h is. By Lemma 3.10, we may choose $X \in \mathrm{Closed}(A \times_B E)$ such that $g := \pi_1|_X$ is minimal. As A is an n-Lee space, it follows from (3) that g is at most n to 1. Clearly, we have the commutative diagram shown below:



$(5) \Rightarrow (3)$: We consider the diagram

$$\begin{array}{ccc} & & E \\ & & \Big\downarrow \rho \\ A & \xrightarrow[\iota_A]{} & A \end{array}$$

with ρ a minimal surjection. Then, using (5), we may complete the square to obtain

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ g \Big\downarrow & & \Big\downarrow \rho \\ A & \xrightarrow[\iota_A]{} & A \end{array}$$

where $g: X \to A$ is at most n to 1 and surjective.

Claim. *f is surjective.*

Proof. We assume, towards a contradiction, that f is not surjective. Hence $f_>(X) \subsetneq E$. But $f_>(X)$ is compact and hence closed.

Also $\rho_>(f_>(X)) = (\rho \circ f)_>(X) = (\iota_A \circ g)_>(X) = g_>(X) = A$.

Hence ρ is not minimal, which is a contradiction. Hence, the claim. □

But $g = \iota_A \circ g = \rho \circ f$. Hence, as f is surjective and g is at most n to 1, so is ρ. We have thus established the exhibited chains of implications. □

Remark 3.12. Part (5) in Theorem 3.11 may be regarded as a generalization of the notion of projectivity, which is recovered as a special case of (5), if g is 1 to 1. Gleason's Theorem can be extended to the category *Top of topological spaces and continuous maps*; the theorem is that the projectives with respect to *proper* surjections in Top are precisely the extremally disconnected spaces. Theorem 3.11 has a similar extension to the category Top of all topological spaces and continuous maps; this is Theorem 3.15 below. Indeed, the notion of propriety as defined in the following was motivated by Gleason's proof of Theorem 3.1 [13, pp. 104–105]. In essence, one analyses the proof to isolate the relevant properties of the maps. We have omitted some details in the proof of the following theorem as it repeats some of the constructions in the preceding one.

Definition 3.13. $f: X \to Y$ is said to be *proper*: $\Leftrightarrow$ f satisfies the following conditions:

(1) for every $y \in Y$, $f^{<}(\{y\})$ is compact.

(2) f is a closed map, i.e. the function $\forall_f : \mathcal{P}(X) \to \mathcal{P}(Y) := O \mapsto (Y \backslash f_>(X \backslash O))$ preserves open sets.

(3) Distinct points in the same fibre of f have disjoint open neighbourhoods in X, or equivalently, the diagonal map $\triangle : X \to X \times_Y X$ is a closed embedding.

Remark 3.14. We note the following relevant properties of proper maps. The proofs may be found in [13, pp. 102–105].

(1) Lemma 3.10 remains valid if we delete the words "compact" and "Hausdorff" and replace the word "continuous" with "proper".

(2) The restriction of a proper map to a closed subspace of its domain is proper.

(3) In the category of topological spaces and continuous maps, pullbacks of proper maps are proper.

Theorem 3.15. *The following are equivalent for every topological space A:*

(1) *A is an n-Lee space.*

(2) *Every proper minimal surjection $\rho : E \to A$ is at most n to 1.*

(3) *Every diagram of the form*

$$\begin{array}{ccc} & & E \\ & & \downarrow h \\ A & \xrightarrow{k} & B \end{array}$$

in Top, with h proper, can be completed to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ g \downarrow & & \downarrow h \\ A & \xrightarrow{k} & B \end{array}$$

where g is at most n to 1.

Proof. (1)$\Rightarrow$(2): We prove this by contraposition. Let a proper minimal map $\rho : E \to A$ be given for which (2) does not hold. Hence, we may choose $a \in A$ such that $\#(\rho^{<}(\{a\})) \geq n+1$. Let $x_1, \ldots, x_{n+1}$ be elements of $\rho^{<}(\{a\})$. We may, using clause (3) in Definition 3.13, choose a family of disjoint open sets $(O_i \mid i \in 1..(n+1))$ such that for every $i \in 1..(n+1)$, $x_i \in O_i$. Set for every $i \in 1..(n+1)$, $H_i := \forall_\rho(O_i)$. As ρ is proper, it follows from clause (2) of Definition 3.13 that for every $i \in 1..(n+1)$, H_i is open. We note that for every pair of distinct elements $i, j \in 1..(n+1)$, $H_i \cap H_j = \forall_\rho(O_i) \cap \forall_\rho(O_j) = \forall_\rho (O_i \cap O_j) = \forall_\rho(\emptyset) = \emptyset$ (by Lemma 3.6).

But, as ρ is minimal, it follows from Proposition 3.7 that for every $i \in 1..(n+1)$, $\rho_{>}(O_i) \subset \mathrm{Cl}(H_i)$. Hence for every $i \in 1..(n+1)$, $\rho(x_i) = a \in \mathrm{Cl}(H_i)$. Hence $\bigcap_{i \in 1..(n+1)} \mathrm{Cl}(H_i) \neq \emptyset$. Hence, A is not an n-Lee space, i.e. (1) does not hold.

(2)$\Rightarrow$(1): We prove this by contraposition. We assume that A is not an n-Lee space. Hence, we may choose $a \in A$ and a family of pairwise disjoint open sets $(O_i \mid i \in 1..(n+1))$ in A such that $a \in \bigcap_{i \in 1..(n+1)} \mathrm{Cl}(O_i)$.

Set $X := \bigcup_{i \in 1..(n+1)} ((A \setminus \bigcup_{j \in I..(n+1) \setminus \{i\}} O_j) \times \{i\})$. $\pi : X \to A$ denotes the projection. We may, by an appropriate modification of Lemma 3.10, choose $E \in \mathrm{Closed}(X)$ such that $\rho := \pi|_E$ is minimal.

Claim. *ρ is proper and not at most n to 1.*

Proof. Let $i \in 1..(n+1)$ be given. The same argument as in the proof of (3) ⇒ (1) in Theorem 3.11 yields that $\rho^{<}(\{a\}) = \{(a,i) \mid i \in 1..(n+1)\}$, i.e. ρ is not at most n to 1. It remains to show that ρ is proper. We first show that π is proper. Clauses (1) and (3) in Definition 3.13 are clearly satisfied. We therefore verify clause (2). Let $C \in \mathrm{Closed}(X)$ be given. Hence we may choose $n+1$ closed sets $C_1, \ldots, C_{n+1} \in \mathrm{Closed}(A)$ such that $C = \bigcup_{i\in 1..(n+1)} C_i \times \{i\}$. Hence $\pi_{>}(C) = \bigcup_{i \in 1..(n+1)} C_i$, which is closed in A. Hence, as C was arbitrary, π is a closed map. Hence, as E is closed, by Remark 3.14, $\rho = \pi|_E$ is proper. Hence the claim. □

Thus, ρ is a minimal proper map that fails to be at most n to 1. Hence the result.

(2) ⇒ (3), (3) ⇒ (2): In view of Remark 3.14, these proofs are essentially the same as the proofs (3) ⇒ (5) and (5) ⇒ (3) in Theorem 3.11 and are therefore omitted.

Thus, we have established the exhibited chain of implications. □

We next consider the validity of the Lee identities in various presheaf topoi. We consider the general case $\mathcal{S}^{\mathrm{Op}(\mathscr{C})}$ first. The background material on presheaf topoi and sieves, as also the interpretation of first-order predicate calculus in topoi, can be found in [17]. Similar considerations in the context of conditions stronger than De Morgan's law appear in [9].

Definition 3.16. Given $C \in \mathrm{Obj}(\mathscr{C})$ and a family of morphisms with the same codomain C, $(f_i \in \mathrm{Cod}^{<}(C) \mid i \in I)$.

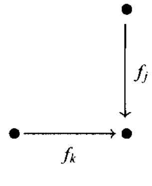
$$\mathrm{Ssp}((f_i \in \mathrm{Cod}^{<}(C) \mid i \in I)) := \text{the sieve spanned by } (f_i \in \mathrm{Cod}^{<}(C) \mid i \in I).$$

Lemma 3.17. *For every $C \in \mathrm{Obj}(\mathscr{C})$ and all sieves R, S on C,*

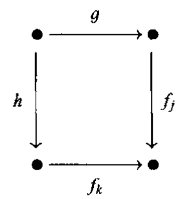
$$\iota_C \in (R \to S) \Leftrightarrow (R \subset S).$$

Proof. $(\iota_C \in (R \subset S)) \Leftrightarrow ((R \to S) = \mathrm{Cod}^{<}(C)) \Leftrightarrow (R = R \cap \mathrm{Cod}^{<}(C) = R \cap (R \to S) \subset S)$. □

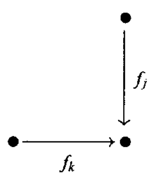
Proposition 3.18. *$\mathcal{S}^{\mathrm{Op}(\mathscr{C})}$ satisfies $I_{n-1} \Leftrightarrow$ given n morphisms $(f_i \mid i \in 1..n)$ with the same codomain C, there exist $j, k \in 1..n$ such that the diagram*



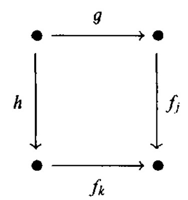
in $\mathscr{C}$ can be completed to a commutative square



Proof. ($\Rightarrow$): Let $C \in \mathrm{Obj}(\mathscr{C})$ and $(f_i \in \mathrm{Cod}^<(C) \mid i \in 1..n)$ be given. If there exist $j,k \in 1..n$ such that $\mathrm{Ssp}(f_j) \cap \mathrm{Ssp}(f_k) \neq \emptyset$, then clearly



may be completed to



for some g and h.

Thus, we assume that for every pair of distinct elements $i,j \in 1..n$, $\mathrm{Ssp}(f_i) \cap \mathrm{Ssp}(f_j) = \emptyset$. As $\mathscr{S}^{\mathrm{Op}(\mathscr{C})}$ satisfies I_{n-1}, it follows that we may choose $i \in 1..n$ such that $\iota_C \in (\mathrm{Ssp}(f_i) \rightarrow \emptyset)$. Hence by Lemma 3.17, $\mathrm{Ssp}(f_i) \subset \emptyset$, which is a contradiction. Hence, the initial assumption is untenable. Hence there exist distinct $j,k \in 1..n$ such that $\mathrm{Ssp}(f_j) \cap \mathrm{Ssp}(f_k) \neq \emptyset$, and we are in the previous case. As C, f_i were arbitrary, the result follows.

($\Leftarrow$): Let $C \in \mathrm{Obj}(\mathscr{C})$ and $(R_i \in \mathrm{Sieve}(C) \mid i \in 1..n)$ be given. We assume

(1) that for every pair of distinct elements $i,j \in 1..n$, $R_i \cap R_j = \emptyset$, and

(2) that the right-hand side of Proposition 3.18 holds.

Assume that

$$\text{for every } i \in 1..n, \quad R_i \neq \emptyset. \tag{2}$$

Then we may choose for every $i \in 1..n$, $f_i \in R_i$. It follows from (1) that for every pair of distinct elements $i,j \in 1..n$, $\mathrm{Ssp}(f_i) \cap \mathrm{Ssp}(f_j) = \emptyset$. But this contradicts (2) i.e. that the antecedent holds. Hence (3) is untenable. Hence we may choose $i \in 1..n$ such that $R_i = \emptyset$. Hence $R_i^* = \mathrm{Cod}^<(C)$. Hence, $\bigcup_{i \in 1..n} R_i^* = \mathrm{Cod}^<(C)$. As C, R_i were arbitrary, it follows that $\mathscr{S}^{\mathrm{Op}(\mathscr{C})}$ satisfies I_{n-1}. $\square$

Remark 3.19. The preceding essentially means that there do not exist n non-empty pairwise disjoint sieves on an object.

Corollary 3.20. *Let M be a monoid. The topos $\mathcal{S}^M$ of (right) M-sets satisfies $I_{n-1} \Leftrightarrow$ the following equivalent conditions hold:*

(1) *There do not exist n non-empty pairwise disjoint right ideals.*

(2) *For every family $(m_i \in M \mid i \in 1..n)$, there exist $j,k \in 1..n$ and $p_j, p_k \in M$ such that $m_j p_j = m_k p_k$. (This is called the right Ore condition if $n=2$ [6].)*

Proof. Direct translation of Proposition 3.18. □

Corollary 3.21. *Let P be a poset. The topos $\mathcal{S}^{\mathrm{Op}(P)}$ satisfies $I_{n-1} \Leftrightarrow$ for every $p \in P$ and $p_1, \ldots, p_n \leq p$ there exist $i,j \in 1..n$ and $r \in P$ such that $r \leq p_1$, and $r \leq p_j$.*

Proof. Direct translation of Proposition 3.18. □

We shall next discuss conditions equivalent to the Lee identities in arbitrary topoi. One has to weaken the notion of primeness to get analogues of Theorem 3.3. We need the following definitions and lemmas before we can state the main results.

Definition 3.22. Let the distributive lattice L be given. We first make the following definitions:

$\mathrm{Idl}(L) := \{I \subset L \mid I$ is an ideal$\}$, $\mathrm{Filt}(L) := \{F \subset L \mid F$ is a filter$\}$, $\mathrm{MaxIdl}(L) := \{I \in \mathrm{Idl}(L) \mid I$ is a maximal ideal$\}$, $\mathrm{MaxFilt}(L) := \{F \in \mathrm{Filt}(L) \mid F$ is a maximal filter$\}$, for every $S \subset L$, $\mathrm{Isp}(S) :=$ the ideal spanned by S, and for every $S \subset L$, $\mathrm{Fsp}(S) :=$ the filter spanned by S.

Lemma 3.23. *Let the distributive lattice L, $I \in \mathrm{Idl}(L)$, $a \in L$, $F \in \mathrm{Filt}(L)$, and $b \in L$ be given. Then*

(1) $\mathrm{Isp}(I \cup \{a\}) = \{(a \wedge k) \vee i \mid i \in I,\ k \in L\}$.

(2) $\mathrm{Fsp}(F \cup \{b\}) = \{(b \vee k) \wedge f \mid f \in F,\ k \in L\}$.

Proof. Straightforward verification. □

Definition 3.24. Let $n \in \omega$ be given.

(1) An ideal I in a ring R is said to be $(n-1)$-*prime*: $\Leftrightarrow \forall a_1 \ldots \forall a_n \in R$, $((\forall i,j \in 1..n,\ i \neq j \Rightarrow a_i a_j \in I) \Rightarrow \exists i \in 1..n,\ a_i \in I)$.

The preceding means that given n elements of R whose pairwise products are all in I, one of these elements must be in I. Similar remarks apply to the following definitions.

(2) An ideal I in a distributive lattice L is said to be $(n-1)$-*prime*: $\Leftrightarrow \forall a_1 \ldots \forall a_n \in L$, $((\forall i,j \in 1..n,\ i \neq j \Rightarrow a_i \wedge a_j \in I) \Rightarrow \exists i \in 1..n,\ a_i \in I)$.

(3) A filter F in a distributive lattice L is said to be $(n-1)$-*prime*: $\Leftrightarrow \forall a_1 \ldots \forall a_n \in L$, $((\forall i,j \in 1..n,\ i \neq j \Rightarrow a_i \vee a_j \in F) \Rightarrow \exists i \in 1..n,\ a_i \in F)$.

Remark 3.25. The preceding definitions are weakenings of the various notions of primeness that we shall need. We note that if $n=2$, we recover the usual notion of primeness. A related definition for n-primeness in the case of ideals in rings was proposed independently by Richard Squire in a similar context [18]. Blass has shown that under the assumption of the axiom of choice, an n-prime ideal in a ring is the intersection of n (1-)prime ideals [3]. It is unknown to the author if this may be established without assuming the axiom of choice.

Definition 3.26. In this definition A is a ring or a distributive lattice in a topos $\mathcal{T}$. An ideal I in A is said to be *proper* $:\Leftrightarrow (1 \in I) \rightarrow \bot$ internally. An ideal I in A is said to be *maximal* $:\Leftrightarrow \forall J \in \Omega^A(((J \text{ is an ideal}) \wedge (I \subset J) \wedge \neg(1 \in J)) \rightarrow (I = J))$ internally.

Fact 3.27. *In what follows, we shall have occasion to use the following facts, which are proved in* [10]. *Let the topos $\mathcal{E}$ be given. Then $\{\bot\}$ and $\{\top\}$ are, respectively, the only proper ideal and the only proper filter in $(\Omega_{\mathcal{E}})_{\neg\neg}$. Both are maximal.*

Theorem 3.28. *The following conditions on a topos $\mathcal{E}$ are equivalent:*

(1) $\mathcal{E} \models I_{n-1}$.

(2) *If A is a commutative ring in $\mathcal{E}$, then every proper maximal ideal of A is internally $(n-1)$-prime.*

(3) *If L is a distributive lattice in $\mathcal{E}$, then every proper maximal ideal of L is internally $(n-1)$-prime.*

(4) *If L is a Boolean algebra in $\mathcal{E}$, then every proper maximal ideal of L is internally $(n-1)$-prime.*

(5) *If L is a distributive lattice in $\mathcal{E}$, then every proper maximal filter of L is internally $(n-1)$-prime.*

(6) *If L is a Boolean algebra in $\mathcal{E}$, then every proper maximal filter of L is internally $(n-1)$-prime.*

Proof. We establish the following chains of implications. $(1)\Rightarrow(2)\Rightarrow(4)$, $(1)\Rightarrow(3)\Rightarrow(4)$, $(4)\Rightarrow(1)$, $(3)\Rightarrow(5)$, $(5)\Rightarrow(6)$, $(6)\Rightarrow(1)$. The proofs of $(1)\Rightarrow(2)$ and $(1)\Rightarrow(3)$ are essentially the same except for notational differences. The proof of $(1)\Rightarrow(5)$ is dual to the proof of $(1)\Rightarrow(3)$ and is omitted.

$(1)\Rightarrow(2)$: Let the commutative ring A in $\mathcal{E}$, $I \in \mathrm{MaxIdl}(A)$ and $a_1,\dots,a_n \in A$ be given such that $\forall i,j \in 1..n,\ i \neq j \Rightarrow a_i a_j \in I$. Set $\forall i \in 1..n,\ J_i := \mathrm{Isp}(I \cup \{a_i\})$. Then $\forall i \in 1..n$, we have $\forall b,\ b \in J_i \Leftrightarrow \exists x, \exists y (y \in I \wedge b = a_i x + y)$. Let $i,j \in n$ be given such that $i \neq j$. Then internally we have the following.

$$
\begin{aligned}
&(1 \in J_i) \wedge (1 \in J_j)\\
&\quad \leftrightarrow \exists x_i, \exists y_i, \exists x_j, \exists y_j (y_i \in I \wedge y_j \in I \wedge y_j \in I \wedge 1 = a_i x_i + y_i = a_j x_j + y_j)\\
&\quad \rightarrow 1 = (a_i x_i + y_i)(a_j x_j + y_j)\\
&\quad \rightarrow 1 = a_i a_j x_i x_j + y_i a_j x_j + y_j a_i x_i + y_i y_j\\
&\quad \rightarrow 1 \in I \ (\text{as } a_i a_j \in I)\\
&\quad \rightarrow \bot \ (\text{as } I \text{ is proper}).
\end{aligned}
$$

As i,j were arbitrary, we conclude that $\forall i,j \in 1..n,\ i \neq j \Rightarrow (1 \in J_i) \wedge (1 \in J_j) = \bot$ Hence, as $\mathscr{E} \models I_{n-1}$, it follows that $\bigvee_{i\in 1..n} \neg(1 \in J_i) = \top$. Hence, as we are arguing in the internal logic, we may choose $i \in 1..n$ such that $\neg(1 \in J_i)$. But, as I is maximal and $I \subset J_i$, it follows that $I = J_i$. Hence $a_i \in I$. As $a_1, \ldots, a_n$ were arbitrary, I is $(n-1)$-prime.

$(1) \Rightarrow (3)$: Let the distributive lattice L in $\mathscr{E}$, $I \in \mathrm{MaxIdl}(L)$ and $a_1, \ldots, a_n \in L$ be given such that $\forall i,j \in 1..n,\ i \neq j \Rightarrow a_i \wedge a_j \in I$. Set $\forall i \in 1..n,\ J_i := \mathrm{Isp}(I \cup \{a_i\})$. Then it follows from Lemma 3.23 that $\forall i \in 1..n$ and $\forall b, b \in J_i \Leftrightarrow \exists x \exists y (y \in I \wedge b = (a_i \wedge x) \vee y)$. Let $i,j \in i..n$ be given such that $i \neq j$. Then internally we have the following:

$$\begin{aligned}
&(1 \in J_i) \wedge (1 \in J_j)\\
&\quad \leftrightarrow \exists x_i \exists y_i \exists x_j \exists y_j (y_i \in I \wedge y_j \in I \wedge 1 = (a_i \wedge x_i) \vee y_i = (a_j \wedge x_j) \vee y_j)\\
&\quad \to 1 = ((a_i \wedge x_i) \vee y_i) \wedge (a_j \wedge x_j) \vee y_j)\\
&\quad \to 1 = (a_i \wedge a_j \wedge x_i \wedge x_j) \vee (y_i \wedge a_j \wedge x_j) \vee (y_j \wedge a_i \wedge x_i) \vee (y_i \wedge y_j)\\
&\quad \to 1 \in I \ \text{(as } a_i \wedge a_j \in I)\\
&\quad \to \bot \ \text{(as } I \text{ is proper)}.
\end{aligned}$$

As i,j were arbitrary, we conclude that $\forall i,j \in 1..n,\ i \neq j \Rightarrow (1 \in J_i) \wedge (1 \in J_j) = \bot$. Hence, as $\mathscr{E} \models I_{n-1}$, it follows that $\bigvee_{i\in 1..n} \neg(1 \in J_i) = \top$. Hence, as we are arguing in the internal logic, we may choose $i \in 1..n$ such that $\neg(1 \in J_i)$. But, as I is maximal and $I \subset J_i$, it follows that $I = J_i$, and hence $a_i \in I$. As $a_1, \ldots, a_n$ were arbitrary, I is $(n-1)$-prime.

$(2) \Rightarrow (4), (3) \Rightarrow (4), (5) \Rightarrow (6)$: $(2) \Rightarrow (4)$ and $(3) \Rightarrow (4)$ follow from the facts that a Boolean algebra is both a commutative ring and a distributive lattice and the notions of ideal, primeness and similarly n-primeness agree in the two contexts [7, pp. 452–457, 12, pp. 10–12]. The identification of a Boolean algebra with a distributive lattice may be made in two ways. One of them sends ideals in the algebra to filters in the lattice. Essentially the same arguments, as in the references cited, yield that n-prime ideals correspond to n-prime filters. $(5) \Rightarrow (6)$ then follows in a manner analogous to $(3) \Rightarrow (4)$.

$(4) \Rightarrow (1)$: We consider the Boolean algebra $\Omega_{\neg\neg}$. The unique proper maximal ideal in $\Omega_{\neg\neg}$ is the singleton $\{\bot\}$. By assumption, $\{\bot\}$ is $(n-1)$-prime. Hence $\forall p_1, \ldots, p_n$ of type $\Omega_{\neg\neg}$ such that $(\forall i,j \in 1..n,\ i \neq j \Rightarrow p_i \wedge p_j = \bot)$, $\exists i \in 1..n$ such that $p_i \in \{\bot\}$ i.e. $p_i = \bot$. Hence,

$$\bigvee_{i\in 1..n} (\neg p_i) = \top. \tag{3}$$

Let $q_1, \ldots, q_n$ of type Ω be given such that $\forall i,j \in 1..n,\ i \neq j \Rightarrow q_i \wedge q_j = \bot$. Then $\neg\neg q_1, \ldots, \neg\neg q_n$ are of type $\Omega_{\neg\neg}$. As Ω is a Heyting algebra, $\forall i,j \in 1..n,\ i \neq j$

$$\Rightarrow \neg\neg q_i \wedge \neg\neg q_j = \neg\neg(q_i \wedge q_j) = \neg\neg(\bot) = \bot.$$

Hence by (1), $\bigvee_{i\in 1..n} \neg(\neg\neg q_i) = \top$. Hence, $\bigvee_{i\in 1..n} \neg(q_i) = \top$. As $q_1, \ldots q_n$ were arbitrary, we conclude that $\mathscr{E} \models I_{n-1}$.

(3) ⇒ (5): This follows immediately from the following observation. If L is a distributive lattice, so is L^{op}, the opposite of L, and so $\text{MaxFilt}(L) = \text{MaxIdl}(L^{op})$ as subobjects of Ω^L. Hence,

$$\begin{aligned} F \in \text{MaxFilt}(L) &\to F \in \text{MaxIdl}(L^{op}) \\ &\to F \text{ is an } (n-1)\text{-prime ideal in } L^{op} \\ &\to F \text{ is an } (n-1)\text{-prime filter in } L. \end{aligned}$$

(6) ⇒ (1): We note that (6) is dual to (4). We consider the Boolean algebra $\Omega_{\neg\neg}$ in $\mathscr{E}$. This has a unique proper filter $\{\top\}$ which is maximal. By assumption, $\{\top\}$ is $(n-1)$-prime. As $\neg : (\Omega_{\neg\neg})^{op} \to \Omega_{\neg\neg}$ is an isomorphism of Boolean algebras, $\{\bot\}$ is the unique proper maximal ideal of $\Omega_{\neg\neg}$ and is $(n-1)$-prime. Then it follows from the proof of (4) ⇒ (1) that $\mathscr{E} \models I_{n-1}$. □

Further development. We list below certain questions closely related to the above that will be considered in a later article. The following results are known.

(1) Let $\mathscr{E}$ be a topos with a natural number object. Then $\mathscr{E} \models I_1 \Leftrightarrow$ the object of Dedekind–Tierney real numbers in $\mathscr{E}$ is (internally) conditionally order-complete [9].

(1) is an internal version of the following result.

(2) Let X be a topological space. Then $\text{Open}(X) \models I_1 \Leftrightarrow$ the set $\text{Cont}(X, \mathbb{R})$ of continuous real-valued functions on X is a conditionally order-complete lattice [4, p. 52].

It remains to find a "nice" lattice-theoretic condition on $\text{Cont}(X, \mathbb{R})$ that corresponds to (2) when I_1 is replaced with I_n. This should immediately generalize to an internal analogue corresponding to (1).

The special case in which the topological space X is the spectrum $\text{Spec}(R)$ of a commutative ring R has also been studied [15, 16]. The following results are known:

(3) $\text{Open}(\text{Spec}(R)) \models I_1 \Leftrightarrow R/N$ is a Baer ring, where N is the nilradical of R.

(4) Let X be a topological space. Then

(i) $\text{Open}(X) \models I_1 \Rightarrow$ the set of continuous real-valued functions on X is a Baer ring, and

(ii) the set of continuous real-valued functions on X is a Baer ring and X is completely regular ⇒ $\text{Open}(X) \models I_1$.

Here, the question is to find ring-theoretic conditions on $\text{Cont}(X, \mathbb{R})$ that generalize (3) and (4) if I_1 is replaced with I_n (with $n \geq 2$). This will be investigated in a forthcoming paper.

Ring-theoretic conditions on $\text{Cont}(X, \mathbb{R})$ equivalent to the validity of I_n in $\text{Open}(\text{Spec}(R))$ for $n \geq 2$ should also yield internal analogues.

In [11, 12] Johnstone has generalized both of Gleason's results, i.e. Theorems 3.1 and 3.2 to the category of topoi and special classes geometric morphisms. To generalize Theorem 3.2 he constructs a De Morgan topos that "best" approximates a topos. Existence of analogous covers for the other Lee identities are being currently investigated. Johnstone's generalization of Theorem 3.1 is the following:

(5) Let $\mathscr{C}$ be an $\mathscr{S}$-topos (and assume Zorn's lemma holds in $\mathscr{S}$). Then $\mathscr{E}$ is projective with respect to CSLC localic morphisms (in particular, with respect to surjective proper morphisms) in $\mathscr{TOP}$ iff it satisfies De Morgan's law i.e. I_1.

Work is underway to find analogues for $n \geq 2$.

Finally Johnstone has also studied the validity of the identity $\forall x, y((x \to y) \vee (y \to x) = 1)$ called strong De Morgan's law in arbitrary topoi [9]. The equational subclasses of algebras satisfying strong De Morgan's law also form a chain [2]. The validity of these equations in topoi lead to similar questions that are also being currently investigated.

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